XXI SemeAd
Seminários em Administração

novembro de 2018
ISSN 2177-3866

ON THE PREDICTABILITY OF HIGH AND LOW PRICES: THE CASE OF BITCOIN

LEANDRO MACIEL
ESCOLA PAULISTA DE POLÍTICA, ECONOMIA E NEGÓCIOS - UNIVERSIDADE FEDERAL DE SÃO PAULO - EPPEN/UNIFESP

ROSANGELA BALLINI
UNIVERSIDADE ESTADUAL DE CAMPINAS (UNICAMP)

ON THE PREDICTABILITY OF HIGH AND LOW PRICES: THE CASE OF BITCOIN

1. INTRODUCTION

The Bitcoin (BTC), as the most popular cryptocurrency traded in the digital money markets, showed a capitalization of about $40.5 billion by mid-2007, representing 89% of the capitalization of all cryptocurrencies[1]. Launched in 2009, Bitcoin transactions are based on an information technology infrastructure and on the lack of a central authority. Instead of relying on central banks, a decentralized computer network validate the transactions and grow money supply of Bitcoin (YERMACK, 2017). Users and investors have perceived huge financial potential in the Bitcoin market, driving the Bitcoin price from US dollar parity in early 2011 to about 1,500 $/BTC in mid-2017. Further, the number of transactions using Bitcoin has increased considerably. According to Polasik et al. (2015), the number of transactions per month using Bitcoin increased from 12,000 to 2.1 million from August 2010 to August 2014 and in December 2015, approximately 200,000 Bitcoin transactions were carried out per day.

Bitcoin particular features, as the absence of a regulatory agency, made the digital money a volatile and speculative currency, resulting in a market quite sensitive to real (e.g., economic, social and political) and fake (e.g., rumours) news. Also, as stated by Alvarez-Ramirez et al. (2018) and Baek and Elbeck (2015), the poorly defined liquidity conditions of the market and the lack of certainty rules for investment realization add fragility to transactions, an effect that is reflected as large price jumps and excessive volatility as compared to traditional currencies and assets. Besides, a literature on cryptocurrencies have emerged, most focused on the legal aspects and underlying blockchain technology.

Some authors, for example, have discussed the efficiency of virtual money markets. Bartos (2015) indicated that the Bitcoin returns follow the hypothesis of efficient markets by showing its fast responses to publicly announced information. On the other hand, Urquhart (2016), based on automatic variance tests, indicated that Bitcoin returns are significantly inefficient over the period from August 1st, 2010 to July 31st, 2016. However, when the sample was split out into two subsample periods, it was found that the Bitcoin market is efficient in the latter period – August 1st, 2013 to July 31st, 2016.

By verifying empirically whether or not the existence of weekly price anomalies, Kurihara and Fukushima (2017) stated that Bitcoin transactions are becoming more efficient, but the Bitcoin returns do not fulfilled the efficient market hypothesis. Additionally, Urquhart (2017) found significant evidence of price clustering at round numbers as a Bitcoin price anomaly.

Bariviera et al. (2017) used detrended fluctuation analysis (DFA) over a sliding window to report that the Hurst exponent changed significantly during the first years of existence of Bitcoin, tending to stabilize from early 2014 to date. Also using DFA, Alvarez-Ramirez et al. (2018) estimated long-range correlations for price returns of Bitcoin. They found that the Bitcoin market exhibits periods of efficiency alternating with periods where the price dynamics are driven by anti-persistence.

The efficiency of Bitcoin market compared to gold, stock and foreign exchange markets is evaluated in the work of Al-Yahyaee et al. (2018). The study found that the long-memory feature and multifractality of the Bitcoin market was stronger and Bitcoin was therefore more inefficient than the gold, stock and currency markets.

Works have also devoted attention to the analysis of Bitcoin volatility. Balcilar et al. (2017), for example, analyzed the causal relation between trading volume and Bitcoin returns and volatility. The causality-in-quantiles test reveals that volume can predict returns – except in Bitcoin bear and bull market regimes. This result highlights the importance of modeling non-linearity and accounting for the tail behaviour when analysing causal relationships between

Bitcoin returns and trading volume.

Katsiampa (2017) explores the optimal conditional heteroskedasticity model with regards to goodness-of-fit to Bitcoin price data. The author showed that the best model is the AR-CGARCH model, highlighting the significance of including both a short-run and a long-run component of the conditional variance. More recently, Lahmiri et al. (2018) investigated the nonlinear patterns of volatility in seven Bitcoin markets. The empirical findings signify the existence of long-range memory in Bitcoin market volatility, irrespectively of distributional inference. The same applies to entropy measurement, which indicates a high degree of randomness in the estimated series.

In general, the recent literature has stated that volatility of Bitcoin prices is an outcome of market sentiments, where the latter can be attributed with the presence of significant 'memory' (KATSIAMPA, 2017; CHEAH et al., 2018; CHEAH & FRY, 2015; LAHMIRI et al., 2018). This emerges from a key element in the determination of Bitcoin prices: the assumption of full confidence of its users. Indeed, the literature has showed that Bitcoin is ideal for risk-averse investors in anticipation of negative shocks to the market (DYHRBERG, 2016a) and could be used as a hedging asset against market specific risk (DYHRBERG, 2016b).

Due to the evidence of long memory of Bitcoin volatility[2], this work aims to investigate whether or not the high and low prices of Bitcoin are predictable and which approach is appropriate to model these prices. Although many research has been devoted to the analysis of the predictability of daily market closing prices, few studies based on econometric time series models examined the case of high and low prices, as for instance the works of Baruník and Dvořáková (2015), Caporin et al. (2013), Cheung et al. (2010), Cheung et al. (2009), He and Hu (2009), and Cheung (2007). Indeed, Caporin et al. (2013) argue that the lack of studies regarding daily high and low asset prices is surprising for at least three reasons: i) the long histories of high and low prices data are readily available; ii) many technical analysis strategies use high and low prices to construct resistance and support levels; iii) these prices can mesure market liquidity and transaction costs.

In particular, daily high and low prices provide valuable information regarding the dynamic process of an asset throughout time. These prices can be seen as references values for investors in order to place buy or sell orders, e.g. through candlestick charts, a popular technical indicator (XIONG et al., 2017; CHEUNG & CHINN, 2001). He and Wan (2009) also stated that the highs and lows are referred to prices at which the excess of demand changes its direction. Additionally, high and low prices are related with the concept of volatility. Alizadeh et al. (2002) show that the difference between the highest and lowest (log) prices of an asset over a fixed sample interval, also known as the (log) range, is a highly efficient volatility measure[3]. Brandt and Diebold (2006) and Shu and Zhang (2006) pointed out that the range-based volatility estimator appears robust to microstructure noise such as bid-ask bounce, which overcomes the limitations of traditional volatility models based on closing prices that fall to use the information contents inside the reference period of the prices, resulting in inaccurate forecasts.

In addition, daily highs and lows can be used as stop-loss bandwidths, providing information about liquidity provisioning and the price discovery process. According to Caporin et al. (2013), high (low) prices are more likely to correspond to ask (bid) quotes; thus, transaction costs and other frictions, such as price discreteness, the tick size (i.e., the minimal increments) or stale prices, might represent disturbing factors. Finally, high and low prices are more likely to be affected by unanticipated public announcements or other unexpected shocks. Therefore, aspects such as market resiliency and quality of the market infrastructure can be determinant (CAPORIN et al., 2013).

Hence, this paper suggests a fractionally cointegrated vector autoregressive model (FC-

VAR), as proposed by Johansen (2008) and Johansen and Nielsen (2010, 2012), to model and predict the relationship between Bitcoin highs and lows. The motivation of this approach is twofold. First, FCVAR modeling is able to capture the cointegrating relationship between high and low prices, i.e. in the short-term they may diverge, but in the long-term they have an embedded convergence path. Second, the range (the difference between high and low prices), as an efficient volatility measure, is assumed to display a long memory, which allows for greater flexibility[4]. As stated by Baruník and Dvořáková (2015), a more general fractional or long-memory framework, where the series are assumed to be integrated of order d and cointegrated of order less than d, i.e. $CI(d-b)$, where $d, b \in \Re$ and, $0 < b \leq d$, is more useful in capturing the empirical properties of data, in accordance on the evidence of long memory in the volatility of Bitcoin returns (KATSIAMPA, 2017; CHEAH et al., 2018; CHEAH & FRY, 2015; LAHMIRI et al., 2018). Therefore, the FCVAR framework has the advantage of modeling both the cointegration between highs and lows, and the long-memory property of the range. The results are then compared against traditional benchmarks over different prediction horizons.

This paper is outlined as follows. Section 2 describes the data and provides a preliminary analysis of daily high and low Bitcoin prices and the range, focusing on their integration, cointegration, and long memory properties. A FCVAR model for high and low Bitcoin prices is presented in Section 3. The predictability analysis is discussed in Section 4. Finally, Section 5 concludes the work and suggests topics for future investigation.

2. ANALYSIS OF DAILY HIGH AND LOW BITCOIN PRICES

This section describes the database and provides an analysis regarding the integration, cointegration and long memory properties of daily high and low Bitcoin prices and their diference, the range. Further, tests for the possible fractional cointegration relationship between highs and lows are also presented.

2.1 Database

The dynamic properties and the predictability of daily high and low Bitcoin prices are investigated considering the period from January 1st, 2012 to February 28th, 2018 within a total of 2,251 observations[5]. Low and high prices of Bitcoin (BTC) to US dollar (USD) rate exchange data were collected from the web site `www.coindesk.com`.

We consider the daily high log-price, $p_t^H = \log(P_t^H)$, the daily low log-price, $p_t^L = \log(P_t^L)$, and the daily range $R_t = p_t^H - p_t^L$, where P_t^H and P_t^L are the high and low prices at t, respectively. Figure 1 shows the BTC/USD exchange rate low and high log-prices for daily frequency. To improve visibility, the daily lows log-prices in Figure 1 are the actual daily low log-prices minus 0.25. The decrease of prices after May 2013 was derived from the failure of Mt. Gox to protect transaction details which also provoked the suspension of trading[6]. From early 2016 to date, the Bitcoin market experienced a significant growth as a result of the subsequent implementation of cryoptosystems to guarantee transaction privacy stabilized the Bitcoin exchange system. Clearly daily highs and lows dynamic suggests the presence of a common trend, indicating that the series are non-stationary and cointegrated. Figure 2 depicts the temporal evolution of the high and low log-prices difference, i.e. the range. It is worth to note that higher values of the range are associated with the periods of high prices variability, confirming its property as a volatility measure.

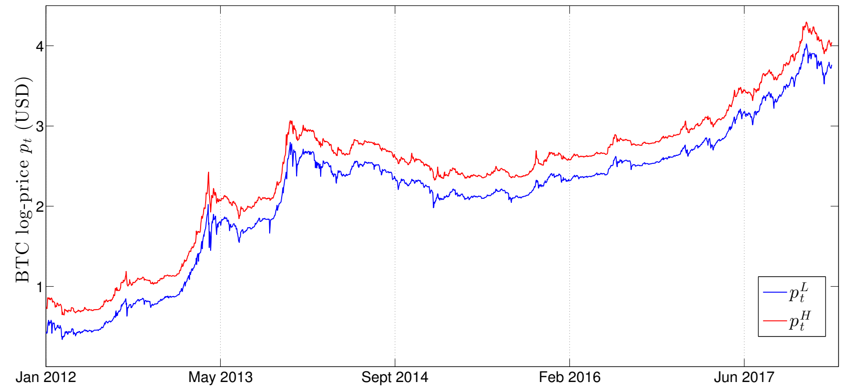

Figure 1. High and low log-prices of Bitcoin to US dollar rate exchange.

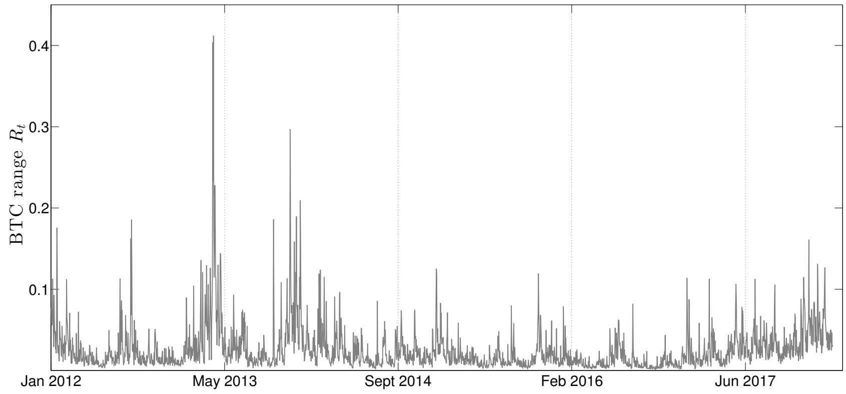

Figure 2. Daily range of Bitcoin.

2.2 Cointegration and memory properties of Bitcoin highs and lows

To analyze the properties of the daily high and low log-prices and the range of Bitcoin, we first evaluate the stationarity of the series. Table 1 provides the Augmented Dickey-Fuller (ADF) (DICKEY & FULLER, 1979) test results for the daily high and low log-prices (p_t^H and p_t^L) as well as the range (R_t), revealing expected findings. Daily high and low prices are unit root processes, i.e. they are non-stationary, under a 0.05 significance level. The daily range is a stationary process, which indicates that daily high and low prices may be cointegrated. Despite these results, it worth to mention that the ADF test is designed to evaluate the null hypothesis of a unit root against the $I(0)$ alternative, i.e. it has very low power against fractional processes.

Table 1. P-values of ADF test for unit root for high (H) and low (L) log-prices and range (R) of Bitcoin based on levels and first-differences, where c denotes the inclusion of a constant only, t the additional inclusion of a trend for daily high and low log-prices in levels only, and lags the number of lags included in the model, based on the Bayesian Information Criteria (BIC) (SCHWARZ, 1978).

Model	Lags	ADF_H		ADF_L		ADF_R
		Level	First-differences	Level	First-differences	Level
c,t	2	0.5847	0.0000	0.6431	0.0001	0.0001

In addition to the ADF test, we performed the KPSS test of Kwiatkowski et al. (1992), appropriate in situations when the tested series are close to being a unit root. The KPSS test results, reported in Table 2, confirm the non-stationarity of the high and low log-prices. However, regarding the range, the results from the KPSS test indicates the presence of a unit root, while the ADF test suggests that the range is stationary. This conflicting results may be caused by the possible long memory property of the range. The results from Table 2 present the KPSS test p-values concerning short lags and long lags in the model. Notice that the results for high and low log-prices for both short and long lags confirm the non-stationarity of the series. On the other hand, when long lags are concerned, the KPSS test results suggest that the range is stationary at a 0.05 significance level. This finding provides evidence on the long memory of the range of Bitcoin prices.

Table 2. P-values of KPSS test for unit root for high (H) and low (L) log-prices and range (R) of Bitcoin based on levels and two lag specifications, short lag and long lag, where c denotes the inclusion of a constant only, t the additional inclusion of a trend for daily high and low log-prices in levels only.

Model	$KPSS_H$		$KPSS_L$		$KPSS_R$	
	Short lag	Long lag	Short lag	Long lag	Short lag	Long lag
c,t	0.0000	0.0001	0.0001	0.0001	0.0001	0.0892*

(*) indicates stationarity at a 0.05 significance level.

Figure 3 shows the autocorrelation function (ACF) of the Bitcoin range. A high degree of persistence is verified, with significance autocorrelations even after 30 lags, confirming the results of the KPSS test and the evidence of long memory of the stock price ranges. This persistence can also be found in the autocorrelation function of the Bitcoin squared range as depicted in Figure 4.

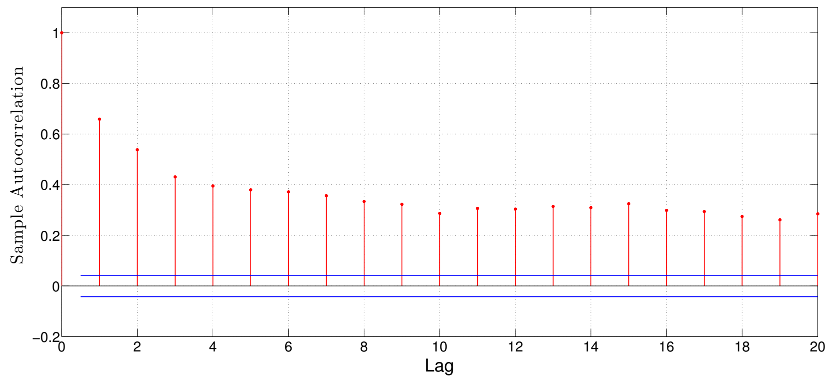

Figure 3. ACF of daily range of Bitcoin.

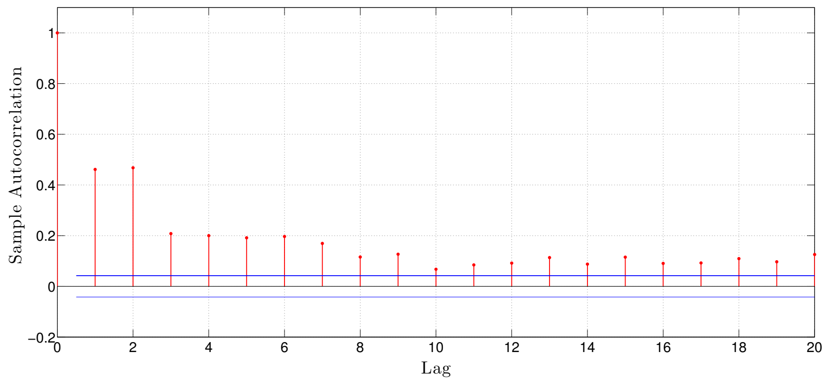

Figure 4. ACF of daily squared range of Bitcoin.

Similar results on the unit root processes of daily high and low asset prices and the stationarity of the range were also found by Cheung (2007(. Therefore, the author suggested a Vector Error Correction Model (VECM) for high and low log-prices. However, due to the high degree of persistence of the range, traditional cointegration analysis may not be satisfactory in explaining the relationship between high and low prices, as already verified by Baruník and Dvořáková (2015) and Caporin et al. (2013), giving rise to the use of the fractionally cointegration framework.

2.3 Testing the fractional cointegration order

The modeling of daily high and low prices as a cointegrated relationship has a particular feature: the "error correction" term, the range, may contain long memory. Differently from Cheung (2007) that used a VECM modeling approach, Baruník and Dvořáková (2015) and Caporin et al. (2013) proposed a fractionally cointegrated model to capture this feature for equity prices. The previous results from Bitcoin high and low prices also suggest the use of the fractional cointegration framework.

Let $X_t \equiv (p_t^H, p_t^L)'$ be a vector composed by the high and low Bitcoin prices, p_t^H and p_t^L, respectively. If the elements of X_t are $I(1)$ and exists a linear combination $\beta' X_t$ that is an $I(0)$ process, X_t is said a cointegrated vector. Robinson and Yajima (2002) indicated that besides the existence of a stable relationship between non-stationary series X_t, i.e. in the short-term they may diverge, but in the long-term they have an embedded convergence path, does not depend on whether the series are $I(1)$. Therefore, to relax the restriction on the choice between stationary $I(0)$ and non-stationary $I(1)$ processes, the series can be considered an $I(d)$ process with $d \in \Re$, where d is the fractional differencing parameter, fractional degree of persistence or fractional order of integration.

The series X_t is an $I(d)$ process if $u_t = (1-L)^d X_t$ is $I(0)$, with L standing for the lag operator and $d < 0.5$ (ROBINSON & YAJIMA, 2002). If $d \geq 0.5$, X_t is defined as a non-stationary $I(d)$ series with $X_t = (1-L)^{-d} u_t I\{t \geq 1\}$, where $t = 0, \pm 1, \pm 2, \ldots$, and $I\{\cdot\}$ is an indicator function. For $d > 0$ ($d < 0$) the process has long-memory (anti-persistence). If $d = 0$, the process collapses to the random walk, i.e. a stationary process.

To test the fractional order of integration of high and low log-prices and the range of Bitcoin, we employed the univariate exact local Whittle (ELW) estimator, as a semi-parametric approach, proposed by Nielsen and Shimotsu (2007). The method is consistent in the presence of absence of cointegration, and also to both stationary and non-stationary cases. The univariate local exact Whittle estimators for high, lows and the range ($\hat{d}^H$, $\hat{d}^L$ and $\hat{d}^R$, respectively) are found by minimizing the following contrast function:

$$Q_{m_d}(d^i, G_{ii}) = \frac{1}{m_d} \sum_{j=1}^{m_d} \left[\log\left(G_{ii} \lambda_j^{-2d^i} \right) + \frac{1}{G_{ii}} I_j \right],\ i = H, L, R, \tag{1}$$

which is concentrated with respect to the diagonal element of the 2×2 matrix G, a finite and nonzero matrix with strictly positive diagonal elements. Under the hypothesis that the spectral density of $U_t = [\Delta^{d^H} p_t^H, \Delta^{d^L} p_t^L, \Delta^{d^R} R_t]$, G satisfies:

$$f_U(\lambda) \sim G \ \text{ as } \ \lambda \to 0, \tag{2}$$

where $f_U(\lambda)$ is the spectral density matrix, I_j the coperiodogram at the Fourier frequency $\lambda_j = \frac{2\pi j}{T}$ of the fractionally differenced series U_t, m_d is the number of frequencies used in the estimation, and T is the sample size (CAPORIN et al., 2013). The matrix G is estimated as:

$$\hat{G} = \frac{1}{m_d} \sum_{j=1}^{m_d} Re(I_j), \tag{3}$$

with $Re(I_j)$ standing for the real part of the coperiodogram.

The estimates of the fractional integration order do not imply the presence or absence of cointegration. To test the equality of integration orders, $H_0: \; d^H = d^L = d$, we also employed the test suggested by Nielsen and Shimotsu (2007), which is robust to the presence of fractional cointegration. In the bivariate case under study, the test statistic is (NIELSEN & SHIMOTSU, 2007):

$$\hat{T}_0 = m_d (S\hat{d})' \left(S \frac{1}{4} \hat{D}^{-1} \left(\hat{G} \odot \hat{G} \right) \hat{D}^{-1} S' + h(T)^2 \right)^{-1} \left(S\hat{d} \right), \tag{4}$$

where $\odot$ is the Hadamard product, $\hat{d} = [\hat{d}^H, \hat{d}^L]$, $S = [1, -1]'$, $h(T) = \log(T)^{-k}$ for $k > 0$, $D = \mathrm{diag}(G_{11}, G_{22})$.

According to Nielsen and Shimotsu (2007), if the variables are not cointegrated, i.e. the cointegration rank is $r = 0$, $\hat{T}_0 \to \chi_1^2$, while if $r \geq 1$, the variables are cointegrated and $T_0 \to 0$. For significant large values of the test statistic $\hat{T}_0$ with respect to the null density χ_1^2, it evidences against the null hypothesis of the equality of integration orders.

The first six columns of Table 3 display the ELW estimates of $\hat{d}^H$, $\hat{d}^L$ and $\hat{d}^R$ for the Bitcoin, where the exponent denotes daily high (H), daily low (L) and daily range (R). The estimates of integration orders were calculated base on two specifications of bandwidth, $m_d = T^{0.5}$ and $m_d = T^{0.6}$, as in the works of Nielsen and Shimotsu (2007), Caporin et al. (2013), and Baruník and Dvořáková (2015). For both bandwidths, the order of integration of daily highs and lows are generally high and close to 1, indicating that the series are not stationary. The difference between high and low prices (the range) is mostly non-stationary ($d > 0$) and displays long memory with parameter $\hat{d}^R < 0.5$, in accordance with the previous findings from the ACF of the range (Figures 3 and 4). Concerning the bandwidth parameter, the results are not significantly sensitive. Summarizing, the daily high and low Bitcoin prices are not stationary and the range displays long memory, in line with the results of Caporin et al. (2013) and Baruník and Dvořáková (2015).

Table 3. Estimates of the fractional order of integration parameter d of high ($\hat{d}^H$) and low ($\hat{d}^L$) log-prices and the range ($\hat{d}^R$) of Bitcoin using the exact local Whittle (ELW) estimator, and test statistics for the equality of integration orders ($\hat{T}_0$). All estimates use both $m_d = T^{0.5}$ and $m_d = T^{0.6}$ as bandwidths.

$\mathrm{ELW}_{m_d=T^{0.5}}$			$\mathrm{ELW}_{m_d=T^{0.6}}$			$\hat{T}_0$	
$\hat{d}^H$	$\hat{d}^L$	$\hat{d}^R$	$\hat{d}^H$	$\hat{d}^L$	$\hat{d}^R$	$m_d = T^{0.5}$	$m_d = T^{0.6}$
0.9524	0.9361	0.3982	0.8973	0.8819	0.4113	0.1207	0.2238

Regarding the test for the equality of integration orders, the last two columns of Table 3 presents the test statistics estimated with $m_d = T^{0.5}$ and $m_d = T^{0.6}$ as bandwidth parameters. Since the critical value of χ_1^2 is 2.71 in a 90% confidence interval, the null hypothesis of equality of the integration orders cannot be rejected for both bandwidth parameters. The results suggest that a FCVAR modeling approach with the same degree of integration orders $d^H = d^L$ is appropriate for estimating the relationship between the daily high and low prices of Bitcoin. Notice that the generalization to the presence of fractional cointegration between highs and lows is novel for the modeling of Bitcoin.

3. FCVAR MODELING FOR DAILY HIGH AND LOW BITCOIN PRICES

The fractionally cointegrated vector autoregression (FCVAR), formalized by Johansen (2008) and Johansen and Nielsen (2010, 2012), generalizes the classical cointegration analysis by allowing X_t to be fractional of order d and cofractional of order $d-b$, which conducts that $\beta' X_t$ should be fractional of order $d-b \geq 0$. This framework allows for the existence of a common stochastic trend, integrated with order d, and the short-term divergences from the long-run equilibrium integrated of order $d-b$. The parameter b is the strength of the cointegrating relationships, called as the cointegration gap (a higher b means less persistence in the cointegrating relationships).

In the FCVAR modeling approach, the usual lag operator and the difference operator are replaced by the fractional lag operator and the fractional difference operator, $L_b = 1-\Delta^b$ and $\Delta^b = (1-L)^b$, respectively (JOHANSEN & NIELSEN, 2012; NIELSEN & MORIN, 2016). The fractional difference operator is defined by the binomial expansion $\Delta^b Z_t = \sum_{n=1}^{\infty} (-1)^n \binom{b}{n} Z_{t-n}$ (BARUNÍK & DVOŘÁKOVÁ, 2015). Thus, the model is applied to $Z_t = \Delta^{d-b} X_t$. A fractionally cointegrated vector autoregressive FCVAR$_{d,b}(p)$ model for $X_t \equiv (p_t^H, p_t^L)'$ as the vector of high and low prices is described as:

$$\Delta^d X_t = \Delta^{d-b} L_b \alpha\beta' X_t + \sum_{i=1}^{p} \Gamma_i \Delta^d L_b^i X_t + \varepsilon, \;\; t = 1, \ldots, T, \tag{5}$$

where α and β are $2 \times r$ matrices comprised by the long-run parameters, $0 \leq r \leq 2$, the rank r is termed the cointegration, or cofractional, rank, $d \geq b > 0$, $\Gamma = (\Gamma_1, \ldots, \Gamma_p)$ are the autoregressive augmentation parameters related to the short-run dynamics, and ε_t is an p-dimensional $i.i.d$ $(0, \Omega)$, with positive-definite variance matrix Ω.

The columns of β constitute the r cointegration (cofractional) vectors such that $\beta' X_t$ are the cointegrating combinations of the variables in the system, i.e. the long-run equilibrium relations. The parameters in α are the adjustment or loading coefficients which represent the speed of adjustment towards equilibrium for each of the variables (NIELSEN & MORIN, 2016). If $d-b < 0.5$, $\beta' X_t$ is asymptotically a zero-mean stationary process. Denoting $\Pi = \alpha\beta'$, where the $2 \times r$ matrices α and β with $r \leq 2$ are assumed to have full column rank r, the columns of β are then the r cointegrating (cofractional) relationship determining the long-run equilibrium.

Non-zero mean data, $Y_t = \mu + X_t$ for example, can be considered as $\Delta^a Y_t = \Delta^a(\mu + X_t) = \Delta^a X_t$, since $\Delta^a 1 = 0$ for $a > 0$. Thus, this means that the model with $d > b$ is invariant to the inclusion of a restricted constant term ρ. As in Baruník and Dvořáková (2015), the inclusion of a constant term is considered only in the model with $d = b$, which replaces the formulation in (5) by:

$$\Delta^d X_t = L_d \alpha(\beta' X_t + \rho') + \sum_{i=1}^{p} \Gamma_i \Delta^d L_b^i X_t + \varepsilon, \;\; t = 1, \ldots, T, \tag{6}$$

where ρ is the restricted constant term $\mu = \alpha\rho'$, interpreted as the mean level of the long-run equilibrium.

The model parameters are estimated by maximum likelihood as described in Nielsen and Morin (2016). Before estimating the FCVAR models for daily high and low prices of Bitcoin, it is required the use of an appropriate approach to test and determine the cointegration rank in the model, described as follows.

3.1 Cointegration rank

Cointegration rank testing in the presence of long memory differs from traditional tests for integration Johansen (1991). A time series X_t is fractionally cointegrated $CI(d,b)$ if X_t has $I(d)$

elements and for some $b > 0$, exists a vector β such that $\beta' X_t$ is integrated of order $(d-b)$. We first applied the cointegration rank test proposed by Nielsen and Shimotsu (2007), that allows for both stationary and non-stationary fractionally integrated processes. The test is based on the exact local Whittle estimate of d, used to examine the rank of the spectral density matrix G and its eigenvalues. In the bivariate case under study, the test estimates the rank r by:

$$\hat{r} = \arg \min_{u=0,1} L(u), \tag{7}$$

where

$$L(u) = v(T)(2-u) - \sum_{i=1}^{2-u} \hat{\delta}_i, \tag{8}$$

for some $v(T) > 0$ which satisfies

$$v(T) + \frac{1}{m_L^{1/2} v(T)} \to 0, \tag{9}$$

with $\hat{\delta}_i$ as the i-th eigenvalue of $\hat{G}$, and m_L a new bandwidth parameter.

The estimation of matrix G involves two steps. First, $\hat{d^H}$ and $\hat{d^L}$ are obtained first using (1) with m_d as bandwidth parameter. Given $\bar{d}_* = (\hat{d^H} + \hat{d^L})/2$, the matrix G is estimated as follows:

$$\hat{G} = \frac{1}{m_L} \sum_{j=1}^{m_L} Re(I_j), \tag{10}$$

such that $m_L/m_d \to 0$. The estimates of G are robust to all different choices of m_d and m_L (NIELSEN & SHIMOTSU, 2007).

Table 4 displays the results of the cointegration rank test of Nielsen and Shimotsu (2007) using $m_d = T^{0.6}$ and $m_L = T^{0.5}$ for both cases where $v(T) = m_L^{-0.45}$ and $v(T) = m_L^{-0.05}$. The results suggest that there is one cointegration relationship. It is worth to note that $L(1) < L(0)$ and this can be taken as strong evidence in favor of fractional cointegration between p_t^H and p_t^L so that the expression in (7) is minimized in correspondence of $r = 1$.

Table 4. Estimates of the fractional cointegration rank test statistics and their respective eigenvalues by the approach of Nielsen and Shimotsu (2007) using $\bar{d}_*$, the average of the estimated integration orders of daily high and low Bitcoin prices from the ELW estimator with $m_d = T^{0.6}$ as bandwidth parameter, in the fractional cointegration analysis for both $v(T) = m_L^{-0.45}$ and $v(T) = m_L^{-0.05}$, with $m_L = T^{0.5}$.

			Rank estimates					
	Eigenvalues		$v(T) = m_L^{-0.45}$			$v(T) = m_L^{-0.05}$		
$\bar{d}_*$	$\hat{\delta}_1$	$\hat{\delta}_2$	$L(0)$	$L(1)$	$\hat{r}$	$L(0)$	$L(1)$	$\hat{r}$
0.9442	0.2982	0.0004	-1.3201	-1.7822	1	-0.4297	-1.2033	1

In addition, the cointegration rank test proposed by Johansen and Nielsen (2012) was also considered. In the FCVAR framework, the hypothesis $H_r : \text{rank}(\Pi) = r$ is tested against the alternative $H_n : \text{rank}(\Pi) = n$. Let $L(d,b,r)$ be the profile likelihood function given rank r, where (α, β, Γ) have been concentrated out by regression and reduced rank regression (NIELSEN & MORIN, 2016). For the model with a constant, the test concerns the hypothesis $H_r : \text{rank}(\Pi, \mu) =$

r against $H_n : \text{rank}(\Pi, \mu) = n$, with $L(d,r)$ as profile likelihood function given rank r, where the parameters $(\alpha, \beta, \rho, \Gamma)$ have been concentrated out by regression and reduced rank regression.

The profile likelihood function is maximized both under the hypothesis H_r and under H_n considering the LR test statistic computed as follows:

$$LR(q) = 2 \log \left(L(\hat{d}_n, \hat{b}_n, n) / L(\hat{d}_r, \hat{b}_r, r) \right), \tag{11}$$

where $q = n - r$ and

$$L(\hat{d}_n, \hat{b}_n, n) = \max_{d,b} L(d,b,n), \text{ and } L(\hat{d}_r, \hat{b}_r, r) = \max_{d,b} L(d,b,r). \tag{12}$$

The asymptotic distribution of $LR(q)$ depends qualitatively (and quantitatively) on the parameter b. In the case of "weak integration", $0 < b < 0.5$, $LR(q)$ has a standard asymptotic distribution (NIELSEN & MORIN, 2016):

$$LR(q) \xrightarrow{\text{D}} \chi^2(q^2), \; 0 < b < 0.5. \tag{13}$$

Otherwise, in the case of "strong cointegration", when $0.5 < b \leq d$, asymptotic theory is nonstandard and

$$LR(q) \xrightarrow{\text{D}} \text{Tr}\left\{ \int_0^1 \mathrm{d}W(s)F(s)' \left(\int_0^1 F(s)F(s)'\mathrm{d}s \right)^{-1} \int_0^1 F(s)\mathrm{d}W(s)' \right\}, \; b \geq 1/2, \tag{14}$$

where the vector process $\mathrm{d}W$ is the increment of ordinary (non-fractional) vector standard Brownian motion of dimension $q = p - r$ (NIELSEN & MORIN, 2016). The vector process F depends on the deterministic in a similar way as in the CVAR model in Johansen (1995). In the model with no determinist term $F(u) = W_b(u)$, otherwise, if the restricted constant term is included in the model, then $F(u) = \left(W_b'(u), 1\right)'$, where $W_b(u) = \Gamma(b)^{-1} \int_0^u (u-s)^{b-1}\mathrm{d}W(s)$ is vector fractional type-II Brownian motion.

Table 5 shows the results of the cointegration test of Johansen and Nielsen (2012) and a significant cointegration relationship was found. For $r = 0$, larger values of the likelihood ratio (LR) statistics indicates the rejection the null hypothesis of zero cointegrating relationship. Otherwise, when $r = 1$, the LR statistics are smaller and the corresponding p-values indicate that we cannot reject the null of one cointegrating relationship.

Table 5. Likelihood ratio (LR) statistics and p-values from the cointegration test by Johansen and Nielsen (2012) for each rank $r = 0,1,2$, as well as the corresponding estimates of the parameter of the fractional order of integration ($\hat{d}$) and the parameter of the cointegration gap ($\hat{b}$) for Bitcoin high and low prices.

$r=0$				$r=1$				$r=2$	
$\hat{d}$	$\hat{b}$	LR	p-value	$\hat{d}$	$\hat{b}$	LR	p-value	$\hat{d}$	$\hat{b}$
0.6892	0.6101	29.9827	0.000	0.9446	0.5987	0.1455	0.7255	0.9972	0.6212

3.2 Empirical FCVAR model for Bitcoin highs and lows

Based on the previous evidence of one significant cointegrating vector for the high and low prices of Bitcoin, a fractionally cointegrating VAR (FCVAR) model was estimated. We set $p = 1$ for the short-term deviations, which is sufficient to capture the autocorrelation of the residuals. Also, as stated by MacKinnon and Nielsen (2014), a single lag is usually sufficient in the fractional model, in contrast with the standard cointegrated VAR where more lags are required to account for the serial correlation in the residuals. The FCVAR model was estimated for the case when $d \neq b$, since all estimates reported earlier rejects the hypothesis where d and b are close to equality (see Table 5).

Table 6 reports the FCVAR estimates for the high and low prices of Bitcoin. Parameters estimates of the fractional integration order and the cointegration gap, $\hat{d}$ and $\hat{b}$ respectively, are significantly different from zero and different from each other. Estimate of $\hat{d}$ indicates that daily high and low prices are integrated of an order close to the unity. Regarding the cointegrating vector, $\hat{\beta}$, the estimates are very close to the vector $[1,-1]$. Since the range is defined as the difference between the high and low daily prices, i.e. $(p_t^H - p_t^L)$, it is expected the cointegrating vector to be $[1,-1]$. The results suggest that a linear combination of the daily high and low prices (the range) is integrated of a non-zero order, and the range is in the stationary region $(d-b < 0.5)$. This finding differs from the one obtained by Baruník and Dvořáková (2015) and Caporin et al. (2013), where the ranges of equities fall mostly in the non-stationary region.

The estimates of the adjustment coefficients, $\hat{\alpha}^H$ and $\hat{\alpha}^L$, which describe the speed of adjustment of p_t^H and p_t^L toward equilibrium, are significantly different from zero (Table 6). Parameter $\hat{\alpha}^H$ is negative and $\hat{\alpha}^L$ is positive, indicating that they move in opposite directions to restore equilibrium after a shock to the system occurs. Considering the absolute value of theses parameters estimates, $\hat{\alpha}^H$ is greater than $\hat{\alpha}^L$, implying that the correction in the equation for daily highs overshoots the long-run equilibrium. These results were also verified by Baruník and Dvořáková (2015) and Caporin et al. (2013) considering the analysis of equity prices, however, in more than 50% of the cases $\hat{\alpha}^H$ estimates were smaller than $\hat{\alpha}^L$.

Table 6. FCVAR model estimates results. Standard errors are shown below the parameters estimates in brackets.

$\hat{d}$	$\hat{b}$	$\hat{\beta}$	$\hat{\alpha}_H$	$\hat{\alpha}_L$	$\hat{\gamma}_{11}$	$\hat{\gamma}_{12}$	$\hat{\gamma}_{21}$	$\hat{\gamma}_{22}$
0.9649	0.5841	[1,-0.9980]	-0.1682	0.0936	-0.0167	0.2562	0.1918	0.1360
(0.0290)	(0.0546)		(0.0200)	(0.0178)	(0.0322)	(0.0382)	(0.0271)	(0.0360)

Concerning the short-run dynamics parameters estimates $\Gamma_1 = (\hat{\gamma}_{11}, \ldots, \hat{\gamma}_{22})$, according to Table 6, the coefficients of the lagged daily highs and lows are mostly positive, which suggests an indication of spill-over effects[7]. Finally, the residuals were also tested for the remaining autocorrelation and heteroskedasticity. In most cases, the null of no autocorrelation was rejected according to the Ljung-Box Q-test, but based on the visualization of the autocorrelation functions, the dependency is weak, and it disappears after the second lag. Some heteroskedasticity was also detected by the autocorrelation function of squared residuals, however, it is very weak.

4. PREDICTABILITY OF DAILY HIGH AND LOW BITCOIN PRICES

Besides the advantages of describing the dynamics of high and low Bitcoin prices and their difference, the range, the forecasting ability of the FCVAR modeling framework is also examined. Forecasts were performed using the FCVAR in an out-of-sample set comprised by the last two years of data. As competing models, we consider the VECM model of Cheung (2007); the random walk, RW; the ARIMA model; the 5-day moving average, MA_5; and the 22-day moving average, MA_{22}; the latter two of which correspond to weekly and monthly averages respectively and are very employed by technical analysts.

The Diebold and Mariano (1995) test is carried out to measure the forecasting superiority of the FCVAR, focusing on the mean squared error (MSE) of the forecasts. The error of the model

i for the h-step ahead forecasting horizon is defined by:

$$\varepsilon^{H}_{t+h,i} = p^{H}_{t+h} - \hat{p}^{H}_{t+h,i}, \tag{15}$$

for the daily high, and

$$\varepsilon^{L}_{t+h,i} = p^{L}_{t+h} - \hat{p}^{L}_{t+h,i}, \tag{16}$$

for the daily low, with $i = \mathrm{FCVAR}, \mathrm{VECM}, \mathrm{RW}, \mathrm{ARIMA}, \mathrm{MA}_5, \mathrm{MA}_{22}$, where p^H_t (p^L_t) and $\hat{p}^H_t$ ($\hat{p}^L_t$) are the actual and predicted high (low) Bitcoin prices at t.

It is worth noting that not only one-step-ahead forecasting is performed to assess the prediction performance of fractionally cointegration models for high and low Bitcoin prices, as made by Caporin et al. (2013) concerning asset prices, but also five- and ten-step-ahead forecasting are performed to examine the medium- and long-term forecasting ability of the empirical FCVAR and selected competitors.

Table 7 shows ranking results of the Diebold and Mariano (1995) test for the out-of-sample forecasts of daily high and low Bitcoin log-prices obtained using the FCVAR against the benchmark models. From the experimental results obtained, the FCVAR approach consistently outperforms all of other competitors (Table 7). The rankings from best to worst are: FCVAR, VECM, ARIMA, RM, MA_5, MA_{22}. As far as the comparison between the FCVAR and VECM, the former presents better results. As expected, the moving average methodologies performed worst. When comparing the performance of each method across the three prediction horizons (i.e., 1, 5, and 10), the superior performance of FCVAR over the remaining methods is still verified. However, predictions of FCVAR and VECM tend to be equally accurate with the increase in prediction horizon. Summing up, the results indicate the predictability of the daily high and low prices of Bitcoin. Moreover, the use of a long memory framework such as the FCVAR do improve forecasting performance in short- and long-term prediction horizons.

Table 7. Forecasting models ranking from Diebold-Mariano test for high and low Bitcoin prices.

Price	Rank of methods										
	1		2		3		4		5		6
Panel A: one-step-ahead prediction horizon											
High	FCVAR	$>^*$	VECM	$>^*$	ARIMA	$>$	RW	$>^*$	MA_5	$>$	MA_{22}
Low	FCVAR	$>^*$	VECM	$>^*$	ARIMA	$>$	RW	$>^*$	MA_5	$>$	MA_{22}
Panel B: five-step-ahead prediction horizon											
High	FCVAR	$>^*$	VECM	$>^*$	ARIMA	$>$	RW	$>^*$	MA_5	$>$	MA_{22}
Low	FCVAR	$>$	VECM	$>^*$	ARIMA	$>$	RW	$>^*$	MA_5	$>$	MA_{22}
Panel C: ten-step-ahead prediction horizon											
High	FCVAR	$>$	VECM	$>^*$	ARIMA	$>$	RW	$>^*$	MA_5	$>$	MA_{22}
Low	FCVAR	$>$	VECM	$>^*$	ARIMA	$>$	RW	$>^*$	MA_5	$>$	MA_{22}

(*) indicates the mean difference between the two competing methods is significant at the 5% level.

Finally, Figure 5 illustrates the performance of FCVAR modeling framework for daily high and low Bitcoin forecasting candlesticks, based on the observed prices with the corresponding predicted high-low bands by FCVAR for the last three months of data considering one-step-ahed predictions. It is interesting to note that FCVAR provide a good fit of the high-low dispersion, indicating the potential of the proposed method which can enhances chart analysis, a tool often used by technical traders.

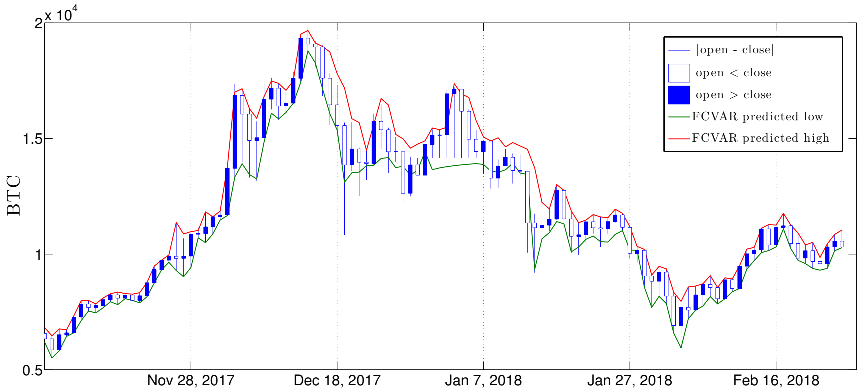

Figure 5. Bitcoin candlesticks and FCVAR predicted high-low bands.

5. CONCLUSION

This work evaluated the predictability and dynamic properties of daily high and low Bitcoin prices. The modeling of daily high and low prices considered a fractionally cointegrated VAR model (FCVAR), which accounts for two fundamental patterns of these prices: their cointegrating relationship and the long-memory of their difference (i.e., the range), as the error correction term is allowed to fall into a non-stationary region. The empirical analysis examined daily high and low prices of Bitcoin (BTC) to US dollar (USD) rate exchange during the period from January 2012 to February 2018. The findings indicate that daily high and low Bitcoin prices are integrated of an order close to the unity, and the range displays long memory and is in the stationary region. A significant cointegration relationship was found between daily high and low prices. The empirical FCVAR model shows that high and low prices move in opposite directions to restore equilibrium after a shock to the system occurs. Also, the results evidence the predictability of daily highs and lows of Bitcoin for different forecasting horizons, in which the fractionally approach conducts to better predictions than competitive methods. Future work shall include the estimation of the FCVAR with the restriction on the cointegrating vector β to be $(1,-1)$, which allows the interpretation of the difference $(d-b)$ as the order of integration of the range. The evaluation of forecasts using an economic criteria, e.g. through a trading strategy, is also demanding and compelling, mainly considering intradaily trading.

Notes

1. Other cryptocurrencies, based on blockchain technology, are for example the Litecoin (LTC), Ethereum (ETH), Ripple (XRP). In the website `https://coinmarketcap.com/currencies/` up to 641 of such monies can be found.

2. The literature have also presented substancial evidence of long memory in the volatility process of asset prices, interest rate differentials, inflation rates, forward premiums and exchange rates (YALAMA & CELIK, 2013; GARVEY & GALLAGHER, 2012; BREIDT et al., 1998; ANDERSEN & BOLLERSLEV, 1997).

3. The literature that considers the high-low range price as a proxy for volatility dates back to the 1980s with the work of Parkinson (1980).

4. The literature considers asset prices to be integrated of order 1, i.e. $I(1)$. However, the choice between stationary, $I(0)$, and non-stationary, $I(1)$, processes can be too restrictive for the

degree of integration of daily high and low prices (BARUNÍK & DVOŘÁKOVÁ, 2015). Since these prices can be considered as a possibly fractionally cointegrated relationship, it improves flexibility, mainly when the error correction term from the cointegrating relationship between high and low prices is the range (CHEUNG, 2007; FIESS & MACDONALD, 2002).

5. We begin the analysis in 2012 since in this period the prices of Bitcoin start to show some relevant volatility.

6. In the early stage (2010–2013), Bitcoin exchange was handled by Mt. Gox, an administration system based on Shibuya, Japan. By the last months of 2013, Mt. Gox was handling about 70% of transactions worldwide.

7. Cheung (2007) states that negative coefficients imply a regressive behavior, whereas positive coefficients are an indication of spill-over effects. In this case, higher daily highs tend to fall to a lower level, lower daily highs tend to drift up to a higher level, and higher daily lows lead to higher daily highs (BARUNÍK & DVOŘÁKOVÁ, 2015).

REFERENCES

AL-YAHYAEE, K. H.; MENSI, W.; YOON, S. Efficiency, multifractality, and the long-memory property of the Bitcoin market: A comparative analysis with stock, currency, and gold markets. **Finance Research Letters**, https://doi.org/10.1016/j.frl.2018.03.017, 2018.

ALIZADEH, S.; BRANDT, M. W.; DIEBOLD, F. X. Range-based estimation of stochastic volatility. **The Journal of Finance**, v. 57, n. 3, pp. 1047–1091, 2002.

ALVAREZ-RAMIREZ, J.; RODRIGUEZ, E.; IBARRA-VALDEZ, C. Long-range correlations and asymmetry in the Bitcoin market. **Physica A**, v. 492, pp. 948–955, 2018.

ANDERSEN, T. G.; BOLLERSLEV, T. Heterogeneous information arrivals and return volatility dynamics: Uncovering the long-run in high frequency returns. **Journal of Finance**, v. 52, n. 3, pp. 975–1005, 1997.

BAEK, C.; ELBECK, M. Bitcoins as an investment or speculative vehicle? A first look. **Applied Economic Letters**, v. 22, n. 1, pp. 30–34, 2015.

BALCILAR, M.; BOURI, E.; GUPTA, R.; ROUBAUD, D. Can volume predict Bitcoin returns and volatility? A quantiles-based approach. **Economic Modelling**, v. 64, pp. 74–81, 2017.

BARIVIERA, A. F.; BASGALL, M. J.; HASPERUÉ, W.; NAIOUF, M. Some stylized facts of the bitcoin market. **Physica A**, v. 484, pp. 82–90, 2017.

BARTOS, J. Does Bitcoin follow the hyphothesis of efficient market? **International Journal of Economic Science**, v. 4, n. 2, pp. 10–23, 2015.

BARUNÍK, J.; DVOŘÁKOVÁ, S. An empirical model of fractionally cointegrated daily high and low stock market prices. **Economic Modelling**, v. 45, pp. 193–206, 2015.

BRANDT, M. W.; DIEBOLD, F. X. A no-arbitrage approach to range-based estimation of return covariances and correlations. **The Journal of Business**, v. 79, n. 1, pp. 61–74, 2006.

BREIDT, F. J.; CRATO, N.; LIMA, P. The detection and estimation of long memory in stochastic volatility. **Journal of Econometrics**, v. 83, n. 1–2, pp. 325–348, 1998.

CAPORIN, M.; RANALDO, A.; MAGISTRIS, P. C. On the predictability of stock prices: A case for high and low prices. **Journal of Banking & Finance**, v. 37, pp. 5132–5146, 2013.

CHEAH, E. T.; FRY, J. M. Speculative bubbles in Bitcoin markets? An empirical investigation into the fundamental value of Bitcoin. **Economics Letters**, v. 30, pp. 32–36, 2015.

CHEAH, E. T.; MISHRA, T.; PARHI, M.; ZHANG, Z. Long memory interdependency and inefficiency in Bitcoin markets. **Economics Letters**, v. 167, pp. 18–25, 2018.

CHEUNG, Y. L.; CHEUNG, Y. W.; HE, A. W.; WAN, A. T. A high-low model of daily stock price ranges. **Journal of Forecasting**, v. 28, pp. 103–119, 2009.

CHEUNG, Y. L.; CHEUNG, Y. W.; WAN, A. T. A trading strategy based on callable bull/bear contracts. **Pacific-Basin Finance Journal**, v. 18, pp. 186–198, 2010.

CHEUNG, Y. W. An empirical model of dialy highs and lows. **International Journal of Finance & Economics**, v. 12, n. 1, pp. 1–20, 2007.

CHEUNG, Y. W.; CHINN, M. D. Currency traders and exchange rate dynamics: A survey of the US market. **Journal of International Money and Finance**, v. 20, n. 4, pp. 439–471, 2001.

DICKEY, D. A.; FULLER, W. A. Distribution of the estimators for autoregressive time series with a unit root. **Journal of the American Statistical Association**, v. 74, n. 266, pp. 427–431, 1979.

DIEBOLD, F. X.; MARIANO, R. S. Comparing predictive accuracy. **Journal of Business and Economics Statistics**, v. 13, n. 3, pp. 253–265, 1995.

DYHRBERG, A. H. Bitcoin, gold and the dollar - A GARCH volatility analysis. **Finance Research Letters**, v. 16, pp. 85–92, 2016a.

DYHRBERG, A. H. Hedging capabilities of Bitcoin. Is it the virtual gold? **Finance Research Letters**, v. 16, pp. 139–144, 2016b.

FIESS, N. M.; MACDONALD, R. Towards the fundamentals of technical analysis: Analysing the information content of high, low and close prices. **Economic Modelling**, v. 19, n. 3, pp. 353–374, 2002.

GARVEY, J. F.; GALLAGHER, L. A. The realised-implied volatility relationship: Recent empirical evidence from FTSE-100 stocks. **Journal of Forecasting**, v. 31, n. 7, pp. 639–660, 2012.

HE, A. W.; WAN, A. T. Predicting dialy highs and lows of exchange rates: A cointegration analysis. **Journal of Applied Statistics**, v. 36, n. 11, pp. 1191–1204, 2009.

HE, L. T.; HU, C. Impacts of interval computing on stock market variability forecasting. **Computational Economics**, v. 33, n. 3, pp. 263–276, 2009.

JOHANSEN, S. Estimation and hypothesis testing of cointegration vectors in Gaussian vector autoregressive models. **Econometrica**, v. 59, n. 6, pp. 1551–1581, 1991.

JOHANSEN, S. **Likelihood-based inference in cointegrated vector autoregressive models**, Oxford University Press, New York, 1995.

JOHANSEN, S. A representation theory for a class of vector autoregressive models for fractional processes. **Econometric Theory**, v. 24, n. 3, pp. 651–676, 2008.

JOHANSEN, S.; NIELSEN, M. O. Likelihood inference for a nonstationary fractional autoregressive model. **Journal of Econometrics**, v. 158, n. 1, pp. 51–66, 2010.

JOHANSEN, S.; NIELSEN, M. O. Likelihood inference for a fractionally cointegrated vector autoregressive model. **Econometrica**, v. 80, n. 6, pp. 2667–2732, 2012.

KATSIAMPA, P. Volatility estimation for Bitcoin: A comparison of GARCH models. **Economics Letters**, v. 158, pp. 3–6, 2017.

KURIHARA, Y.; FUKUSHIMA, A. The market efficiency of Bitcoin: A weekly anomaly perspective. **Journal of Applied Finance and Banking**, v. 7, pp. 57–64, 2017.

KWIATKOWSKI, D.; PHILLIPS, P. C. B.; SCHMIDT, P.; SHIN, Y. Testing the null hypothesis of stationarity against the alternative of a unit root. **Journal of Econometrics**, v. 54, n. 1, pp. 159–178, 1992.

LAHMIRI, S.; BEKIROS, S.; SALVI, A. Long-range memory, distributional variation and randomness of bitcoin volatility. **Chaos, Solitons and Fractals** v. 107, pp. 43–48, 2018.

MACKINNON, J. G.; NIELSEN, M. O. Numerical distribution functions of fractional unit root and cointegration tests. **Journal of Applied Econometrics**, v. 29, n. 1, pp. 161–171, 2014.

NIELSEN, M.; MORIN, L. A Matlab program and user's guide for the fractionally cointegrated VAR model. **Technical report**, QED Working Paper 1330. Queen's University, 2016.

NIELSEN, M. O.; SHIMOTSU, K. Determining the cointegrating rank in nonstationary fractional systems by the exact local Whittle approach. **Journal of Econometrics** v. 141, n. 2, pp. 574–596, 2007.

PARKINSON, M. The extreme value method for estimating the variance of the rate of return. **The Journal of Business**, v. 53, n. 1, pp. 61–65, 1980.

POLASIK, M.; PIOTROWSKA, A. I.; WISNIEWSKI, T. P.; KOTKOWSKI, R.; LIGHTFOOT, G. Price fluctuations and the use of bitcoin: An empirical inquiry. **International Journal of Electronic Commerce**, v. 20, n. 1, pp. 9–49, 2015.

ROBINSON, P. M.; YAJIMA, Y. Determination of cointegrating rank in fractional systems. **Journal of Econometrics**, v. 106, n. 2, pp. 217–241, 2002.

SHU, J. H.; ZHANG, J. E. Testing range estimators of historical volatility. **Journal of Future Markets**, v. 26, n. 3, pp. 297–313, 2006.

URWUHART, A. The inefficiency of Bitcoin. **Economics Letters**, v. 148, pp. 80–82, 2016.

URQUHART, A. Price clustering in Bitcoin. **Economics Letters**, v. 159, pp. 145–148, 2017.

XIONG, T.; LI, C.; BAO, Y. Interval-valued time series forecasting using a novel hybrid Holt and MSVR model. **Economic Modelling**, v. 60, pp. 11–23, 2017.

YALAMA, A.; CELIK, S. Real or spurious long memory characteristics of volatility: Empirical evidence from an emerging market. **Economic Modelling**, v. 30, pp. 67–72, 2013.

YERMACK, D. Corporate governance and blockchains. **Review of Finance**, v. 21, pp. 7–31, 2017.